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AN ENERGY EXTREMUM PRINCIPLE FOR LOCAL ACTION†

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Abstract—This work describes the solution in terms of the energy of the mixed boundary problem, formulated for the elastic body subjected to prescribed boundary displacements field. The extremum theorems herein proved are particular corollaries of the classical reciprocity theorems. Let us consider a part, ∂V_P, of the unconstrained boundary containing point **P** on which a displacement field $\mathbf{u}_P$ shall be prescribed. The displacement is produced by tractions acting on a part, ∂V_Q, of the unconstrained boundary containing a point **Q** and disjoined from ∂V_P. The strain energy of the body in this elastic state is greater than the strain energy produced by boundary forces acting on ∂V_P and creating the same displacement field $\mathbf{u}_P$ there. The lower bound theorem herein proved gives a quantification of Boussinesque's local perturbation principle and a measure of the strain energy related to local action. The theorem applies both for structures and solids.

1. INTRODUCTION

The local perturbation principle introduced by Boussinesque (1885) and Love (1944) is very important in the theory of elasticity. It states that the effect produced by a field of forces applied in the neighbourhood of a point **Q** of an elastic body rapidly decreases as we move away from point **Q**. The principle of local perturbation perfectly fits the fundamental solution introduced by Kelvin and Boussinesque for unbounded elastic bodies, and the foundation of the assumption of asymptotic regularity formulated by Kellog (1954) for the solution of potential problems related to the presence of a point source. The Saint Venant equivalence principle describes the particular application of Boussinesque's principle to cylindrical bodies.

This work proves an extremum principle for strain energy applied to a class of elastic states of linearly elastic bodies, characterized by prescribed values for displacement field $\mathbf{u}_P$ on the points of a limited boundary region ∂V_P containing a point **P**, generated by force fields acting on a portion ∂V_Q. For this purpose we shall use the concept of Green's function or influence function of the displacement by applied forces.

The forces producing the prescribed displacement field shall be continuous tractions applied to regular parts ∂V_Q of the unconstrained boundary of the body. In the case of non-regular actions it is assumed that both the tractions and the displacements belong to the space $L_2(\partial V)$ of the square integrable functions over ∂V; the ∂V_P and ∂V_Q parts of ∂V must be of Lipschitz type, i.e. the cone of the normals on each point exists. In the case of structures which a regular and invertible influence function for concentrated forces can be defined for, i.e. in those cases where the problem can be formulated in terms of generalized forces and displacements in a one-to-one invertible relation the theorem is valid and shows a more immediate algebraic deduction.

2. THE MINIMUM THEOREM FOR LOCAL ACTION IN STRUCTURES

For the sake of a clearer presentation, the deduction of the theorem in the case of generalized actions is introduced first (Minutolo and Nunziante, 1992). Elastic systems are taken into consideration, whose formulation allows the definition of a continuous displacement influence function for concentrated loads. The concentrated load introduced in this paragraph is meant as a generalized action, i.e. as a resultant of the actions distributed on given areas. For the sake of simplicity we shall be refering to fixed constraint structures.

† This work is dedicated to the memory of Professor Vincenzo Franciosi.

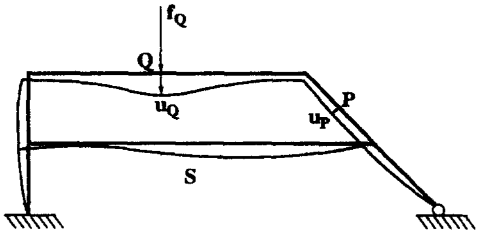

Fig. 1.

An elastic structure S (Fig. 1) is examined, which the influence function $\mathbf{G}_{PQ}$ for the displacement on point **P** by effect of a concentrated force applied on **Q** is defined for. $\mathbf{G}_{PQ}$ has a matrix feature and is regularly symmetrical and biunivocal by assumption for any pair of points **P** and **Q** belonging to S. Therefore $\mathbf{G}_{PQ}$ must be a symmetrical and positively defined matrix.

The force $\mathbf{f}_Q \neq \mathbf{0}$ acting on **Q** produces the displacement $\mathbf{u}_P$ on **P** which is given by:

$$\mathbf{u}_P = \mathbf{G}_{PQ}\mathbf{f}_Q \neq \mathbf{0}. \tag{1}$$

Moreover on its application point **Q** the force $\mathbf{f}_Q$ produces a displacement equal to:

$$\mathbf{u}_Q = \mathbf{G}_{QQ}\mathbf{f}_Q$$

which does not vanish.

The strain energy of the structure S can be obtained through Clapeyron's Theorem and is given by the following relation:

$$W_Q = \tfrac{1}{2}\mathbf{f}_Q \cdot \mathbf{u}_Q = \tfrac{1}{2}\mathbf{f}_Q \cdot \mathbf{G}_{QQ}\mathbf{f}_Q > 0. \tag{2}$$

At this point a force $\mathbf{f}_P$ acts on point **P** of the body which produces a displacement in **P** opposite to the one $\mathbf{u}_P$ given by eqn (1), and thus equal to $[-\mathbf{u}_P]$ (Fig. 2).

The force $\mathbf{f}_P$ to be applied on **P** to obtain the displacement $-\mathbf{u}_P$ can be calculated through the influence function $\mathbf{G}_{PP}$ by means of:

$$\mathbf{f}_P = -\mathbf{G}_{PP}^{-1}\mathbf{u}_P.$$

This elastic state corresponding to the force $\mathbf{f}_P$ acting on **P** is associated with the strain energy, the value of which is:

$$W_P = -\tfrac{1}{2}\mathbf{u}_P \cdot \mathbf{f}_P = \tfrac{1}{2}\mathbf{u}_P \cdot \mathbf{G}_{PP}^{-1}\mathbf{u}_P. \tag{3}$$

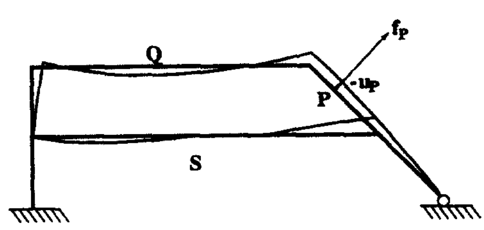

Fig. 2.

If both forces $\mathbf{f}_Q$ and $\mathbf{f}_P$ above act on the structure, the following displacement values are obtained on points **P** and **Q**:

$$\bar{\mathbf{u}}_P = \mathbf{G}_{PP}\mathbf{f}_P + \mathbf{G}_{PQ}\mathbf{f}_Q = \mathbf{0},$$
$$\bar{\mathbf{u}}_Q = \mathbf{G}_{QP}\mathbf{f}_P + \mathbf{G}_{QQ}\mathbf{f}_Q,$$

and the strain energy for such an elastic state of the structure is given by:

$$\begin{aligned} W_{PQ} &= \tfrac{1}{2}\{\mathbf{f}_P\cdot\bar{\mathbf{u}}_P + \mathbf{f}_Q\cdot\bar{\mathbf{u}}_Q\} = \tfrac{1}{2}\mathbf{f}_Q\cdot\bar{\mathbf{u}}_Q \\ &= \tfrac{1}{2}\{\mathbf{f}_Q\cdot\mathbf{G}_{QQ}\mathbf{f}_Q + \mathbf{f}_Q\cdot\mathbf{G}_{QP}\mathbf{f}_P\}. \end{aligned} \tag{4}$$

Bearing in mind that the term $\mathbf{G}_{PQ}\mathbf{f}_Q$ represents the displacement $\mathbf{u}_P$ in eqn (1), opposite that produced by $\mathbf{f}_P$ and that the influence function **G** is symmetrical, the second term on the right-hand side of (4) can be rewritten as follows:

$$\tfrac{1}{2}\mathbf{f}_Q\cdot\mathbf{G}_{QP}\mathbf{f}_P = \tfrac{1}{2}\mathbf{f}_P\cdot\mathbf{G}_{PQ}\mathbf{f}_Q = -W_P. \tag{5}$$

Comparing (4) with (2), (3) and (5) we obtain the following expression for the strain energy W_{PQ}:

$$W_{PQ} = W_Q - W_P. \tag{6}$$

Due to the non-negativeness of W_{PQ} we actually have:

$$W_Q > W_P. \tag{7}$$

It can therefore be maintained that the strain energy of an elastic linear homogeneous material structure subjected to a system of two concentrated forces applied on two points **P** and **Q** and producing null resultant displacement on **P**, is given by the difference between the strain energies related to separately applied systems of forces.

On the other hand the strain energy W_P related to displacement $-\mathbf{u}_P$ in eqn (3) due to $\mathbf{f}_P$, is equal to the one corresponding to the displacement $\mathbf{u}_P$ generated by the force $-\mathbf{f}_P$ acting on **P**. At this point eqn (7) takes on the meaning as expressed in the following:

Minimum theorem. The strain energy necessary to produce a given displacement $\mathbf{u}_P$ *on a point* **P** *on the structure through a concentrated force acting on any other point* **Q** *presents its proper minimum corresponding to the application of a force exactly on* **P**.

3. THE LOWER BOUND THEOREM FOR LOCAL ACTION ON BODIES

For the purpose of applying part of the results above to a body treated according to the modern elasticity theory, next a linearly elastic body V with a suitably regular boundary ∂V is taken into consideration according to the usual assumptions (Sternberg and Eubanks, 1955; Villaggio, 1977; Gurtin, 1973). The boundary ∂V shall be partitioned into ∂V_f and ∂V_u. Forces shall be allocated on ∂V_f. For the sake of simplicity, fixed constraints shall be considered on ∂V_u:

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{x}\in\partial V_u.$$

As already mentioned, the boundary ∂V is considered to be duly regular, so as to give a sense to the integral equations introduced. Moreover we assume that all parts of ∂V taken into consideration have a non-null measure.

Let's consider a point **Q** and a subset ∂V_Q of free boundary ∂V_f containing **Q**. On the part $\partial V_Q \subset \partial V_f$ a system of tractions $\mathbf{f}_Q \neq \mathbf{0}$ square integrable over ∂V_Q is applied; on $(\partial V_f \backslash \partial V_q)$ tractions shall be null. Suppose there exist Green's functions for forces applied

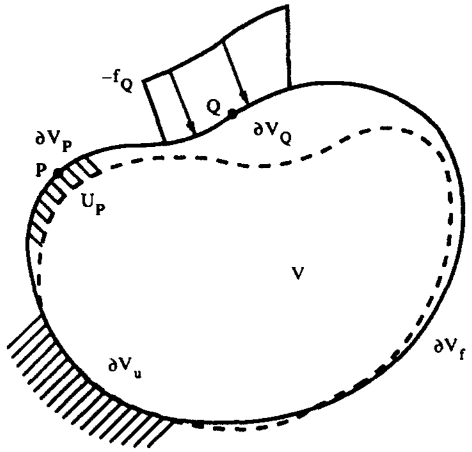

Fig. 3.

on $\mathbf{x} \in V$ of the displacement of the point $\mathbf{y} \in V$ which will be denoted by $\mathbf{g}(\mathbf{y}, \mathbf{x})$. By hypothesis it represents a continuous solution in $(V \backslash \{\mathbf{x}\})$ of Navier's equation for vanishing volume forces and respects the boundary conditions over ∂V_u.

Through function $\mathbf{g}$ it is possible to determine the displacement field of point $\mathbf{y} \in V$ produced by tractions $\mathbf{f}_Q$ acting on ∂V_Q as follows (Fig. 3):

$$\mathbf{u}(\mathbf{y}) = \int_{\partial V_Q} \mathbf{g}(\mathbf{y}, \mathbf{x}) \mathbf{f}_Q(\mathbf{x}) \, \mathrm{d}S(\mathbf{x}), \quad \mathbf{y} \in V, \mathbf{x} \in \partial V. \tag{8}$$

Because the introduced functions are square integrable it is possible to calculate the strain energy of the body V under the prescribed forces, which is given by:

$$W_Q = \frac{1}{2} \int_{\partial V_Q} \mathbf{f}_Q(\mathbf{y}) \left[\int_{\partial V_Q} \mathbf{g}(\mathbf{y}, \mathbf{x}) \mathbf{f}_Q(\mathbf{x}) \, \mathrm{d}S(\mathbf{x}) \right] \mathrm{d}S(\mathbf{y}) \tag{9}$$

and is positive $\forall \, \mathbf{f} \neq \mathbf{0}$.

The following is a more compact form of eqn (9):†

$$W_Q = \tfrac{1}{2}(\mathbf{f}_Q, \mathbf{G}\mathbf{f}_Q). \tag{10}$$

Let's now examine a part ∂V_P of the boundary ∂V_f containing point $\mathbf{P}$, disjoined from ∂V_Q, and let's apply on ∂V_P a field of forces $\mathbf{f}_P$ defined as follows (Fig. 4):

$$\mathbf{f}_P(\mathbf{x}) \neq \mathbf{0}, \quad \mathbf{x} \in \partial V_P,$$
$$\mathbf{f}_P(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \{\partial V_f \backslash \partial V_P\}.$$

This field of forces produces the following displacement in V:

$$\mathbf{u}(\mathbf{y}) = \int_{\partial V_P} \mathbf{g}(\mathbf{y}, \mathbf{x}) \mathbf{f}_P(\mathbf{x}) \, \mathrm{d}S(\mathbf{x}) = \mathbf{G}\mathbf{f}_P, \quad \mathbf{y} \in V, \mathbf{x} \in \partial V_P, \tag{11}$$

the strain energy associated to this elastic solution is:

$$W_P = \tfrac{1}{2}(\mathbf{f}_P, \mathbf{G}\mathbf{f}_P). \tag{12}$$

† The symbol $(\cdot, \cdot)$ stands for the inner product between elements of force and displacement spaces. The symbol $\mathbf{Gf}$ indicates the linear transformation as defined by (8) which $\mathbf{g}(\mathbf{y}, \mathbf{x})$ operates on $\mathbf{f}$.

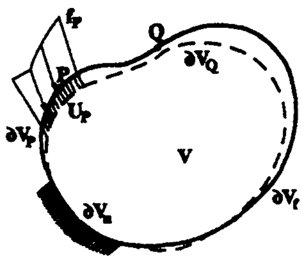

Fig. 4.

Equation (11) for a prescribed field of forces $\mathbf{f}_P$ defines a displacement field $\mathbf{u}$ in the body under investigation. If instead, the displacement field $\mathbf{u}(\mathbf{y}) \in L_2(\partial V_P)$ not vanishing on ∂V_P is prescribed, since the kernel $\mathbf{g}(\mathbf{y}, \mathbf{x})$ of this equation is symmetrical and $\mathbf{g} \in L_2(\partial V_P)$, eqn (11) becomes a Fredholm equation of the first kind in the unknown $\mathbf{f}_P$.

Let's assume that the problem under investigation admits a solution $\mathbf{f}_P^*$: The $\mathbf{f}_P^*$ belongs to space L_2 and it is the only solution to eqn (11), provided the kernel $\mathbf{g}$ is complete in ∂V_P (Smirnov, 1992). The completeness of the kernel is assured if the homogeneous equation associated to (11)

$$\int_{\partial V_P} \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{f}_P \, dS(\mathbf{x}) = \mathbf{0}$$

admits only the trivial solution.

With reference to the elastic problem implying the action of forces $-\mathbf{f}_Q$ on the part ∂V_Q, (8) gives the displacement field $\mathbf{u}_P$ on the part ∂V_P :

$$\mathbf{u}_P = -\mathbf{G}\mathbf{f}_Q. \tag{13}$$

Let's now take into consideration the second elastic problem concerning forces applied only on ∂V_P, able to produce the displacement given by (13) on ∂V_P. In consideration of the above, the inverted form of (11) gives the forces $\mathbf{f}_P^*$ which in this case need to be applied on ∂V_P :

$$\mathbf{f}_P^* = \mathbf{G}^{-1}\mathbf{u}_P. \tag{14}$$

For the two elastic problems as defined above, through (13) and (14) we obtain :

$$-\int_{\partial V_Q} \mathbf{g}(\mathbf{y}, \mathbf{x}) \mathbf{f}_Q(\mathbf{x}) \, ds(\mathbf{x}) = \int_{\partial V_P} \mathbf{g}(\mathbf{y}, \mathbf{x}) \mathbf{f}_P^*(\mathbf{x}) \, ds(\mathbf{x}). \tag{15}$$

The strain energy related to the second of problems introduced above is expressed by :

$$W_P = \tfrac{1}{2}(\mathbf{f}_P^*, \mathbf{u}_P) = -\tfrac{1}{2}(\mathbf{f}_P^*, \mathbf{G}\mathbf{f}_Q) > 0. \tag{16}$$

Finally, let's consider the third elastic problem where the tractions respect the following relations :

$$\mathbf{f} = \begin{cases} \mathbf{f}(\mathbf{x}) = \mathbf{f}_Q(\mathbf{x}) \neq \mathbf{0}, & \mathbf{x} \in \partial V_Q, \\ \mathbf{f}(\mathbf{x}) = \mathbf{f}_P^*(\mathbf{x}) \neq \mathbf{0}, & \mathbf{x} \in \partial V_P, \\ \mathbf{f}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \partial V_f \backslash \{\partial V_P \cup \partial V_Q\}. \end{cases} \tag{17}$$

We immediately notice that the displacement field produced by the application of tractions $\mathbf{f}$ presents identically vanishing values on ∂V_P ; this problem is associated with the strain energy given by :

$$W = \frac{1}{2}(\mathbf{f},\mathbf{u}) = \frac{1}{2}\int_{\partial V} \mathbf{f}(\mathbf{y}) \cdot \left[\int_{\partial V} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}(\mathbf{y})\, \mathrm{d}S(\mathbf{x}) \right] \mathrm{d}S(\mathbf{y})$$

$$= \frac{1}{2}\int_{\partial V_Q} [\mathbf{f}(\mathbf{y})] \cdot \left[\int_{\partial V_P} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_P^*(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) + \int_{\partial V_Q} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_Q(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \right] \mathrm{d}S(\mathbf{y}). \quad (18)$$

The term in square brackets in (18) represents the displacement on ∂V_Q produced by the application of the two fields of forces $\mathbf{f}_Q$ and $\mathbf{f}_P^*$.

If we perform the decomposition by summation of the integral in (18), we obtain the following expression:

$$W = \frac{1}{2}\left[\int_{\partial V_Q} \mathbf{f}_Q(\mathbf{y}) \int_{\partial V_P} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_P^*(\mathbf{x})\, \mathrm{d}S(\mathbf{x})\, \mathrm{d}S(\mathbf{y}) + \int_{\partial V_Q} \mathbf{f}_Q(\mathbf{y}) \int_{\partial V_Q} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_Q(\mathbf{x})\, \mathrm{d}S(\mathbf{x})\, \mathrm{d}S(\mathbf{y}) \right]. \quad (19)$$

The symmetry of the kernel $\mathbf{g}(\mathbf{y}, \mathbf{x})$ allows us to recognize that the first addendum in (19) has a value equal to $-W_P$:

$$\frac{1}{2}\int_{\partial V_Q} \mathbf{f}_Q(\mathbf{y}) \left[\int_{\partial V_P} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_P^*(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \right] \mathrm{d}S(\mathbf{y})$$

$$= \frac{1}{2}\int_{\partial V_P} \mathbf{f}_P^*(\mathbf{y}) \left[\int_{\partial V_Q} \mathbf{g}(\mathbf{y},\mathbf{x})\mathbf{f}_Q(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \right] \mathrm{d}S(\mathbf{y}) = -W_P, \quad (20)$$

and that the second addendum in (19) has a value W_Q.

We obtain:

$$W = W_Q - W_P. \quad (21)$$

Since the left-hand side of (21) must be positive, we obtain the pursued lower bound for W_Q

$$W_Q > W_P. \quad (22)$$

Similar considerations to those at the end of the previous paragraph apply to the case dealt with in this one. Therefore (22) takes the following meaning:

Lower bound theorem. With reference to the elastic solutions characterized by displacement fields having a prescribed value $\mathbf{u}_P$ *on the part* $\partial V_P \subset \partial V_f$ *of the free boundary and generated by suitable fields of tractions acting on any part* $\partial V_Q \subset \partial V_f$ *disjoined from* ∂V_P, *the energy value* W_P *constitutes a lower bound of the strain energy* W_Q *above; where* W_P *corresponds to the elastic problem requesting the application of the tractions on* ∂V_P *and generating the same displacements field* $\mathbf{u}_P$ *there.*

This work allows us to tackle some of the issues (Sternberg and Eubanks, 1955; Maier, 1991) concerning the limit case of concentrated forces acting on the boundary of the body.

4. CONCLUSIONS

The above results are particular corollaries of classical reciprocity theorems. The extremum theorems might be seen in relation to the criteria of asymptotic convergence of the discretized form of some types of boundary integral equations concerning elastic problems

(Guarracino *et al.*, 1991). It is noteworthy that the well-conditioned leading diagonal of the matrix originates from the prevalence of the strain energy related to the collocation force.

Finally, by means of the energy theorems above, it could be possible to state that the properties of positivity and symmetry of operators which don't exist in the Boundary Elements theory, hold definitively when the mesh improves.

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